ELSEVIER Journal of Pure and Applied Algebra 116 (1997) 221–248

JOURNAL OF PURE AND APPLIED ALGEBRA

Relations in operational categories

J. Lambek*

McGill University, 805 Sherbrooke Street, Montreal, PQ, Canada H3A2K6

Received 4 January 1996; revised 21 June 1996

To Peter Freyd

Abstract

There are two well-known methods for constructing the 'connecting homomorphism' in homological algebra. Either one constructs it as a binary relation which is shown to be universally defined and single-valued, or one makes use of the so-called two-square lemma, which provides an isomorphism between two invariants associated with adjacent commutative squares. Both constructions generalize to arbitrary 'Goursat categories', namely operational categories satisfying the condition that the relative product of any relation with its converse is transitive. By an 'operational category' we here mean a category $\mathscr{C}$ accompanied by a category of set-valued functors from $\mathscr{C}$. In order to state the results, one has to define 'relations' in operational categories and one has to generalize the notion of 'exactness' from short sequences to forks and the notion of 'commutativity' to squares in which two arrows are doubled. The proof of the general two square lemma involves a construction which closely resembles that of PER in theoretical computer science and contains the latter as a special case. © 1997 Elsevier Science B.V.

1991 Math. Subj. Class.: 18B10

1. The connecting homomorphism

Ever since functions were declared to be universally defined and single valued, other binary relations have been pushed into the background of mathematics. A notable exception is the construction of the so-called *connecting homomorphism* in homological algebra. Consider the solid part of the diagram in Fig. 1 in a concrete Abelian category, say in Mod R.

* Email: lambek@triples.math.mcgill.ca.

PII S0022-4049(96)00167-3

$$\begin{tikzcd}
 & & & & 0 \arrow[d] & & \\
 & & & A \arrow[r] \arrow[d] & B \arrow[r, dotted, "e"] \arrow[d, "f"] & P \arrow[r, dotted] & 0 \\
 & & C \arrow[r] \arrow[d] & D \arrow[r, "g"] \arrow[d, "h"] & E \arrow[r] \arrow[d] & 0 & \\
 & 0 \arrow[r] & F \arrow[r, "i"] \arrow[d, "j"] & G \arrow[r] \arrow[d] & H & & \\
0 \arrow[r, dotted] & Q \arrow[r, dotted, "k"] & I \arrow[r] \arrow[d] & J & & & \\
 & & 0 & & & &
\end{tikzcd}$$

Fig. 1.

Theorem 1.1. *Assume that all (solid) rows and columns of Fig.* 1 *are exact and that all squares commute. Then there exists an arrow $\varphi : B \to I$ such that $A \to B \to I \to J$ is exact.*

Proof. Complete the diagram with the help of the dotted arrows. Let k^v denote the converse of k. Then the zig-zag homomorphic relation

$$\psi = k^v j i^v h g^v f e^v$$

is an isomorphism $P \xrightarrow{\sim} Q$. Put $\varphi = k\psi e$. $\square$

With the help of the Freyd–Mitchell embedding theorem one may extend this result to other Abelian categories. How does it generalize to non-Abelian categories? In order not to lose sight of applications, we shall phrase our result in terms of 'operational categories'.

Theorem 1.2. *In any* operational category, *assume that all (solid) rows and columns of Fig.* 2 *are* exact, *that all squares* quasi-commute, *that f and i are* injections *and that g and j are* surjections. *Assume further that the solid diagram can be completed by the dotted arrows to render the top and bottom rows exact. Then there exists an arrow $\varphi : B \to I$ such that $A \rightrightarrows B \to I \rightrightarrows J$ is exact. In a* Goursat category, *we may take φ to be the* homomorphic relation

$$\varphi = k k^v j i^v h g^v f e^v e.$$

Before proving this, in Sections 7 and 9, we must define the words in italics: *operational category, exact, quasi-commute, injection, surjection, Goursat category* and *homomorphic relation*. There remains the question when the diagram can be completed as assumed. A sufficient condition will be given in Section 5.

$$\begin{tikzcd}
 & & A \arrow[r, shift left] \arrow[r, shift right] \arrow[d, shift left] \arrow[d, shift right] \arrow[dr, phantom, "(1)"] & B \arrow[r, dotted, "e"] \arrow[d, "f"] & P \\
 & C \arrow[r, shift left] \arrow[r, shift right] \arrow[d, shift left] \arrow[d, shift right] \arrow[dr, phantom, "(3)"] & D \arrow[r, "g"] \arrow[d, "h"] \arrow[dr, phantom, "(2)"] & E \arrow[d, shift left] \arrow[d, shift right] & \\
 & F \arrow[r, "i"] \arrow[d, "j"] \arrow[dr, phantom, "(4)"] & G \arrow[r, shift left] \arrow[r, shift right] \arrow[d, shift left] \arrow[d, shift right] & H & \\
Q \arrow[r, dotted, "k"] & I \arrow[r, shift left] \arrow[r, shift right] & J & &
\end{tikzcd}$$

Fig. 2.

2. Operational categories

An *operational category* $\mathscr{C}_{\mathscr{G}}$ consists of a category $\mathscr{C}$, an auxiliary category $\mathscr{G}$, the category of *sorts*, and a bifunctor

$$[-,-] : \mathscr{G} \times \mathscr{C} \to \text{Set}.$$

In the special case when $\mathscr{G}$ is a category of functors $\mathscr{C} \to$ Set, this bifunctor may be taken to be *application* and we may write

$$[G, C] = G(C).$$

For our purposes, we may as well make the *blanket assumption* that we are dealing with this special case.

However, in many examples, $\mathscr{G}$ is a subcategory of $\mathscr{C}^{\text{op}}$, $\mathscr{C}$ being locally small, and $[-,-]$ is then the usual Hom-functor restricted to $\mathscr{G} \times \mathscr{C}$. In order to subsume these examples under our blanket assumption, we shall identify each object C of $\mathscr{C}^{\text{op}}$ with its image

$$\text{Hom}(C,-) : \mathscr{C} \to \text{Set}$$

under the Yoneda embedding.

Given objects A and B of $\mathscr{C}$, a *potential morphism* $\varphi : A \to B$ is a natural transformation

$$\varphi : [-,A] \to [-,B].$$

It is called a *morphism* if it is representable by an arrow $f : A \to B$ in $\mathscr{C}$, i.e. if $\varphi = [-,f]$. To assure the uniqueness of f, it will be useful to stipulate the following.

Postulate I. $\mathscr{G}$ *generates* $\mathscr{C}$, i.e. if $f, g : A \rightrightarrows B$ and $[G,f] = [G,g]$ for all G in $\mathscr{G}$, then $f = g$.

In case $\mathscr{G} \subseteq \mathscr{C}^{\text{op}}$, this is the usual meaning of 'generates'. If f represents φ, we will just write $\varphi = f$, so that

$$f_G = [G,f] = G(f).$$

Given any set I, by an *I-ary operation* we shall understand a natural transformation

$$\omega : \prod_{i \in I} G_i \to G,$$

where the G_i and G are sorts, i.e. objects of $\mathscr{G}$, but the product

$$P = \prod_{i \in I} G_i$$

is taken in the functor category $\mathrm{Set}^{\mathscr{C}}$.

We shall call the potential morphism $\varphi : A \to B$ a *homomorphism* if it *preserves* all operations, i.e. if the following square commutes for each operation $\omega : P \to G$:

$$\begin{tikzcd} P(A) \arrow[r, "\omega(A)"] \arrow[d, "\varphi_P"'] & G(A) \arrow[d, "\varphi_G"] \\ P(B) \arrow[r, "\omega(B)"'] & G(B) \end{tikzcd}$$

Here φ_P is the unique mapping making the following squares commute for all $i \in I$:

$$\begin{tikzcd} P(A) \arrow[r, "\pi_i(A)"] \arrow[d, "\varphi_P"'] & G_i(A) \arrow[d, "\varphi_{G_i}"] \\ P(B) \arrow[r, "\pi_i(B)"'] & G_i(B) \end{tikzcd}$$

where the $\pi_i : P \to G_i$ are the canonical projections. Note that

$$P(A) = \prod_{i \in I} G_i(A)$$

in Set with canonical projections $\pi_i(A)$, in view of the definition of products in functor categories. If P happens to be in $\mathscr{G}$, the meaning of φ_P is what one would expect.

Of course, every morphism is a homomorphism, with $\varphi_G = G(f)$ and $\varphi_P = P(f)$. It will be useful to stipulate conversely:

Postulate II. Every homomorphism is a morphism.

Then, in view of Postulate I, every homomorphism $\varphi : A \to B$ is *represented* by a unique arrow $f : A \to B$ in $\mathscr{C}$ and we write $\varphi = f$. Postulate II is satisfied, e.g., if $\mathcal{G} \subseteq \mathscr{C}^{\mathrm{op}}$ and $\mathscr{G}^{\mathrm{op}}$ is *adequate* in the sense of Isbell [12].

3. Homomorphic relations

Given an operational category $\mathscr{C}_{\mathscr{G}}$, with $\mathscr{G} \subseteq \mathrm{Set}^{\mathscr{C}}$, by a *potential relation* $\rho : A \not\rightarrow B$ between objects A and B of $\mathscr{C}$ we shall understand a *natural family* of ordinary relations

$$\{\rho_G : G(A) \not\rightarrow G(B)\}_{G \in \mathscr{G}},$$

meaning that, for any natural transformation $\tau : G \to G'$ and any $a \in G(A)$ and $b \in G(B)$,

$$b \rho_G a \Rightarrow \tau(B)(b) \rho_{G'} \tau(A)(a).$$

A potential relation will be called a *homomorphic relation* if it *preserves* all operations

$$\omega : P = \prod_{i \in I} G_i \to G,$$

i.e. if, for all $a \in P(A)$ and $b \in P(B)$,

$$\forall_{i \in I} b_i \rho_{G_i} a_i \Rightarrow \omega(B)(b) \rho_G \omega(A)(a).$$

Here $a_i \in G_i(A)$ is the image of $a \in P(A)$ under the projection $P(A) \to G_i(A)$, i.e.

$$a_i = \pi_i(A)(a).$$

Special homomorphic relations are the following:

(1) homomorphisms $\varphi = [\__{\mathscr{G}}, f]$, when $f : A \to B$ in $\mathscr{C}$ and

$$b \varphi_G a \Leftrightarrow b = f_G(a)$$

for all $a \in [G, A]$, $b \in [G, B]$, G in $\mathscr{G}$ (we usually just write $\varphi = f$);

(2) the *converse* $\rho^{\upsilon} : B \not\rightarrow A$ of a homomorphic relation $\rho : A \not\rightarrow B$, where $\rho^{\upsilon}_G = (\rho_G)^{\upsilon}$, i.e.

$$a \rho^{\upsilon}_G b \Leftrightarrow b \rho_G a;$$

(3) the *relative product* $\sigma\rho : A \not\rightarrow C$ of two homomorphic relations $\rho : A \not\rightarrow B$ and $\sigma : B \not\rightarrow C$, where $(\sigma\rho)_G = \sigma_G \rho_G$ is the usual relative product, i.e.

$$c(\sigma\rho)_G a \Leftrightarrow \exists_{b \in [G,B]}(c \sigma_G b \wedge b \rho_G a);$$

(4) any intersection of homomorphic relations $A \not\rightarrow B$.

By (1)–(3) all *zig-zag* relations are homomorphic:

$$\rho = f g^{\upsilon} h k^{\upsilon} \cdots$$

We shall say that a homomorphic relation ρ is *representable* if $\rho = f g^{\upsilon}$. In case $\mathscr{G} \subseteq \mathscr{C}^{\mathrm{op}}$, all zig-zag relations will be representable under the very mild assumption that $\mathscr{C}$ has *weak pullbacks*. In fact, to say that $B \xleftarrow{p} D \xrightarrow{q} A$ is a weak pullback of $B \xrightarrow{g} C \xleftarrow{h} A$ means that, for all $a : G \to A$ and $b : G \to B$,

$$gb = ha \Leftrightarrow \exists_{d : G \to D}(b = pd \wedge qd = a).$$

If G is in $\mathscr{G}$, this may be written

$$(g^{\upsilon}h)_G = (pq^{\upsilon})_G.$$

This is so for all G in $\mathscr{G}$ if and only if $g^{\upsilon}h = pq^{\upsilon}$.

The crux of this argument may be summarized as follows.

Proposition 3.1. *If $\mathscr{G} \subseteq \mathscr{C}^{\mathrm{op}}$ then $B \xleftarrow{p} D \xrightarrow{q} A$ is a weak pullback of $B \xleftarrow{g} C \xrightarrow{h} A$ if and only if $g^{\upsilon}h = pq^{\upsilon}$.*

If $\rho, \sigma : A \not\to B$, we write $\rho \leq \sigma$ to mean that $b\rho_G a \Rightarrow b\sigma_G a$ for all $a \in [G,A]$, $b \in [G,B]$ and G in $\mathscr{G}$. We say that a potential relation $\rho : A \not\to B$ is

single valued if $\rho\rho^{\upsilon} \leq 1_B$,
universally defined if $1_A \leq \rho^{\upsilon}\rho$,
injective if $\rho^{\upsilon}\rho \leq 1_A$,
surjective if $1_B \leq \rho\rho^{\upsilon}$.

A homomorphic relation which is single valued and universally defined is of course a homomorphism. In particular, for any arrow $f : A \to B$, we have

$$ff^{\upsilon} \leq 1_B, \qquad 1_A \leq f^{\upsilon}f,$$

from which it follows that $ff^{\upsilon}f = f$, $\quad f^{\upsilon}ff^{\upsilon} = f^{\upsilon}$.

4. Injections and surjections

An *injection* is an injective homomorphism, a *surjection* is a surjective homomorphism. The following criterion is useful.

Proposition 4.1. *The representable homomorphic relation fg^{υ} is a homomorphism if and only if $g^{\upsilon}g \leq f^{\upsilon}f$ and g is a surjection. The* corepresentable *homomorphic relation $q^{\upsilon}p$ is a homomorphism only if $qq^{\upsilon} \geq pp^{\upsilon}$; the converse holds if q is an injection.*

Proof. We shall prove the first statement, leaving the second to the reader. Let $\rho = fg^{\upsilon}$, where $f : C \to B$ and $g : C \to A$, say. Then ρ is single valued if and only if $fg^{\upsilon}gf^{\upsilon} \leq 1_B$, i.e. $g^{\upsilon}g \leq f^{\upsilon}f$. Moreover, ρ is universally defined if and only if $1_A \leq gf^{\upsilon}fg^{\upsilon}$, i.e.

$$\forall_{a \in [G,A]}\exists_{c,c' \in [G,C]}(a = g_G(c) \wedge f_G(c) = f_G(c') \wedge g_G(c') = a),$$

i.e.

$$\forall_{a\in[G,B]}\exists_{c\in[G,C]}a = g_G(c),$$

i.e.

$$1_A \leq gg^v. \quad \square$$

Corollary 4.2. *If g is a surjection, then $g^v g \leq f^v f$ if and only if $\exists_{h:A\to B} hg = f$. If q is an injection then $qq^v \geq pp^v$ if and only if $\exists_h qh = p$.*

Proof. Again, we only prove the first statement. If $hg = f$, then surely $f^v f = g^v h^v hg \geq g^v g$, since $h^v h \geq 1_A$. Conversely, if $g^v g \leq f^v f$, then fg^v is a homomorphism by Proposition 4.1, so put $fg^v = h$, in view of (II). But then $hg = fg^v g \geq f$, since $g^v g \geq 1_C$, therefore $hg = f$. $\square$

Proposition 4.3. *Every injection is a monomorphism. The converse is true if $\mathscr{G} \subseteq \mathscr{C}^{\mathrm{op}}$.*

Proof. Suppose $m : A \to B$ is injective, i.e. $m^v m \leq 1_A$. Assume that $mf = mg$, where $f,g : C \rightrightarrows A$. Then $mf \leq mg$, hence $f \leq m^v mg \leq g$ and therefore $f = g$. Thus m is a monomorphism.

For m to be injective means that, for all G in $\mathscr{G}$ and $a, a' \in [G,A]$, $f_G(a) = f_G(a')$ implies $a = a'$. This is surely so when f is a monomorphism and G is in $\mathscr{C}$, for then $f_G(a) = fa$. $\square$

We recall that an epimorphism is said to be *regular* if it is the coequalizer of some pair of parallel arrows. An object P of $\mathscr{G}$ *preserves regular epis* if, for any regular epimorphism $e : A \to B$, every arrow $f \in [P,B]$ can be 'lifted' to an arrow $g \in [P,A]$ so that $[P,e](g) = f$. If P is in $\mathscr{C}$, it is then called *projective*.

Proposition 4.4. *Every surjection is an epimorphism; in case $\mathscr{G} \subseteq \mathscr{C}^{\mathrm{op}}$, if it has a weak kernel pair it is even a regular epimorphism, namely the coequalizer of that pair. Every regular epimorphism is surjective if and only if every object G of $\mathscr{G}$ preserves regular epis.*

Proof. Let $e : B \to C$ be a surjection and assume that $fe = ge$, where $f,g : C \rightrightarrows D$. Then $fe \leq ge$, hence $f \leq fee^v \leq gee^v \leq g$ and therefore $f = g$. Thus e is an epimorphism.

Suppose $\mathscr{G} \subseteq \mathscr{C}^{\mathrm{op}}$ and e has a weak kernel pair $f,g : A \rightrightarrows B$. Then (f,g) is a weak pullback of (e,e), hence $fg^v = e^v e$, as follows from the Proposition 3.1. Now suppose $hf = hg$. Then $e^v e = fg^v \leq h^v h$. Therefore, by Corollary 4.2, there exists $k : C \to D$ such that $ke = h$. This k is unique, since e is an epimorphism. Thus, e is the coequalizer of (f,g).

The last statement of Proposition 4.4 is just a rewording of what it means for G to preserve regular epis. $\square$

5. Exactness

A potential relation θ *on* A (meaning $\theta : A \not\rightarrow A$) is said to be

reflexive if $1_A \leq \theta$,
symmetric if $\theta^v \leq \theta$,
transitive if $\theta\theta \leq \theta$;

it is a potential *equivalence relation* if it has all three properties. Among homomorphic equivalence relations on A are those of the form $h^v h$, where $h : A \to B$, they are called *congruences*.

The potential relation θ is said to be *coreflexive* if $\theta \leq 1_A$. We do not need the concept 'cosymmetric', since θ is symmetric if and only if $\theta^v = \theta$. Nor do we need to define 'cotransitive' for a potential relation which is symmetric, for it is transitive if and only if $\theta\theta = \theta$. Thus, a *coequivalence* may be defined as a transitive and symmetric potential relation which is coreflexive. Among coequivalences on A are those of the form hh^v, where $h : C \to A$ for some object C, they will be called *cocongruences*.

Proposition 5.1. *A representable relation is a cocongruence if and only if it is coreflexive. A corepresentable relation is a congruence if and only if it is reflexive.*

Proof. To prove the first statement, for example, suppose $fg^v \leq 1_A$. Then $f \leq g$, hence $f = g$. □

We note that in a 'Barr exact' category all representable equivalence relations are congruences by definition.

We shall say that the *left fork* $A \underset{g}{\overset{f}{\rightrightarrows}} B \xrightarrow{h} C$ is *exact* if $h^v h$ is the intersection of all congruences on B containing fg^v. This is equivalent to saying

$$\forall_{k : B \to D}(fg^v \leq k^v k \Leftrightarrow h^v h \leq k^v k)$$

for all objects D of $\mathscr{C}$. Now

$$fg^v \leq k^v k \Leftrightarrow kf \leq kg \Leftrightarrow kf = kg.$$

So we may infer the following from Corollary 4.2.

Proposition 5.2. *If h is a surjection, the left fork is exact if and only if h is the coequalizer of (f, g)*

We shall write

$$\text{Ker } h = h^v h,$$

$$\text{Im}(f, g) = \text{intersection of all congruences on } B \text{ containing } fg^v.$$

Exactness of the left fork may thus be expressed by saying that $\mathrm{Im}(f,g)$ equals $\mathrm{Ker}\, h$. In view of Proposition 5.2, $\mathrm{Im}(f,g)$ will be a congruence if (f,g) has a surjective coequalizer.

We shall say that the *right fork* $C \xrightarrow{h} B \overset{f}{\underset{g}{\rightrightarrows}} A$ is *exact* if hh^v is the join of all cocongruences on B contained in $g^v f$. This is equivalent to saying

$$\forall_{k:D\to B}(g^v f \geq kk^v \Leftrightarrow hh^v \geq kk^v)$$

for all objects D of $\mathscr{C}$. Again, we may rewrite $g^v f \geq kk^v$ as $fk = gk$, so we may infer from Corollary 4.2.

Proposition 5.3. *If h is an injection, the right fork is exact if and only if h is the equalizer of (f,g).*

We shall write

$$\mathrm{Im}\, h = hh^v,$$

$$\mathrm{Ker}(f,g) = \text{join of all cocongruences on } B \text{ contained in } g^v f.$$

Exactness of the right fork may thus be expressed by saying that $\mathrm{Ker}(f,g)$ equals $\mathrm{Im}\, h$. In view of Proposition 5.3, $\mathrm{Ker}(f,g)$ is a cocongruence if (f,g) has an equalizer.

Proposition 5.4. *If $\mathscr{C}$ has a terminal object 1, then $e:A \to B$ is surjective if and only if $A \to B \rightrightarrows 1$ is exact. If $\mathscr{C}$ has an initial object 0, then $m:A \to B$ is an injection if and only if $0 \rightrightarrows A \to B$ is exact.*

Proof. We shall prove the first statement, leaving the second to the reader. Let $o_B : B \to 1$ be the unique arrow. Then exactness means that, for all $k : C \to B$,

$$o_B^v o_B \geq kk^v \Leftrightarrow ee^v \geq kk^v.$$

Now $o_B^v o_B \geq 1_B \geq kk^v$, so exactness just asserts that ee^v is the largest cocongruence relation on B, that is, that $ee^v = 1_B$. □

We are now in a position to answer the question: when can the diagram of Theorem 1.2 be completed by the dotted arrows?

Proposition 5.5. *For the diagram of Theorem 1.2 to be completable by the dotted arrows so that top and bottom forks are exact, it suffices that $\mathscr{C}$ has equalizers and coequalizers and that $\mathscr{G}^{\mathrm{op}} \subseteq \mathscr{C}$ consists entirely of projectives.*

Proof. The coequalizer $e:B \to P$ of $A \rightrightarrows B$ is a surjection, by Proposition 4.4, and renders the top fork exact, by Proposition 5.2. The equalizer $k:G \to I$ of $I \rightrightarrows J$ is an injection, by Proposition 4.3, and renders the bottom fork exact, by Proposition 5.3. □

6. Maltsev and Goursat categories

We note that in any operational category cocongruence relations *permute*:

$$ff^{v}gg^{v} = gg^{v}ff^{v}.$$

(For reasons of symmetry, $=$ may be replaced by $\leq$.)

Indeed, $b(ff^{v}gg^{v})_G b'$ means that

$$b = f_G(a), \qquad f_G(a) = b'', \qquad b'' = g_G(a'), \qquad g_G(a') = b'$$

for some $a, a' \in [G, A]$ and $b'' \in [G, B]$, that is,

$$b = f_G(a) = g_G(a') = b',$$

for some $a, a' \in [G, A]$, where now the roles of f and g are interchangeable.

We shall say that an operational category is a *Maltsev*$_1$ *category* if congruence relations permute:

$$f^{v}fg^{v}g = g^{v}gf^{v}f.$$

For reasons of symmetry, we may replace $=$ by $\leq$, the result being equivalent to

$$gf^{v}fg^{v}gf^{v} \leq gf^{v},$$

that is, to

$$(*) \qquad \rho\rho^{v}\rho = \rho$$

for any representable homomorphic relation ρ, since always $\rho\rho^{v}\rho \geq \rho$. If we require $(*)$ for all zig-zag relations or even for all homomorphic relations, we shall speak of a *Maltsev*$_2$ or a *Maltsev*$_3$ category, respectively.

An operational category is a *Goursat*$_1$ category if congruences 3-*permute*:

$$f^{v}fg^{v}gf^{v}f = g^{v}gf^{v}fg^{v}g.$$

Arguing as above, we see that this is equivalent to

$$(**) \qquad \rho\rho^{v}\rho\rho^{v} = \rho\rho^{v}$$

for any representable ρ. Requiring $(**)$ for all zig-zag relations or all homomorphic relations, we obtain *Goursat*$_2$ or *Goursat*$_3$ categories, respectively.

7. Quasi-commutativity

In operational categories we are dealing with two kinds of partially doubled squares, as follows:

$$\begin{tikzcd} A \arrow[r, shift left] \arrow[r, shift right] \arrow[d, shift left] \arrow[d, shift right] & B \arrow[d, "f"] \\ D \arrow[r, "g"'] & E \end{tikzcd}, \qquad \begin{tikzcd} B \arrow[r, "g"] \arrow[d, "f"'] & C \arrow[d, shift left] \arrow[d, shift right] \\ E \arrow[r, shift left] \arrow[r, shift right] & F \end{tikzcd}$$

In a *Goursat*$_2$ category, we shall say that the first of these *quasi-commutes* if

$$ff^v g \ \mathrm{Im}(A \rightrightarrows D) g^v f f^v = g g^v f \ \mathrm{Im}(A \rightrightarrows B) f^v g g^v,$$

and that the second square *quasi-commutes* if

$$f^v f g^v \ \mathrm{Ker}(C \rightrightarrows F) g f^v f = g^v g f^v \ \mathrm{Ker}(E \rightrightarrows F) f g^v g.$$

While these definitions lack motivation for the moment, we may point out already that, in a module category, quasi-commutativity is implied by the usual commutativity of squares, provided all double arrows are replaced by their differences, (see Section 10 below).

The following two *kite lemmas* will prove useful in diagram chasing:

Lemma 7.1. *Consider the kite-shaped diagram in a* Goursat$_2$ *category*:

$$\begin{tikzcd} A \arrow[r, shift left] \arrow[r, shift right] \arrow[d, shift left] \arrow[d, shift right] & B \arrow[r, "e"] \arrow[d, "f"] & P \\ D \arrow[r, "g"] \arrow[d, "h"] & E & \\ G & & \end{tikzcd}$$

Assume that the square quasi-commutes and that the top row and left column are exact. Then

$$hg^v f e^v e f^v g h^v = hg^v f f^v g h^v.$$

Proof. Since the top row is exact and since $gg^vg = g$, the left-hand side of the equation may be written

$$hg^v g g^v f \ \mathrm{Im}(A \rightrightarrows B) f^v g g^v g h^v.$$

Since the square quasi-commutes, this is same as

$$hg^v f f^v g \ \mathrm{Im}(A \rightrightarrows D) g^v f f^v g h^v.$$

Since the left column is exact, this may be written

$$hg^{\vee}ff^{\vee}gh^{\vee}hg^{\vee}ff^{\vee}gh^{\vee}.$$

Finally, applying the Goursat condition to $\rho = hg^{\vee}f$, we obtain

$$hg^{\vee}ff^{\vee}gh^{\vee}. \qquad \square$$

Lemma 7.2. *Consider the kite-shaped diagram in a* Goursat$_2$ *category*:

$$\begin{tikzcd}
 & & B \arrow[d, "f"] \\
 & D \arrow[r, "g"] \arrow[d, "h"] & E \arrow[d, shift left] \arrow[d, shift right] \\
F \arrow[r, "i"] & G \arrow[r, shift left] \arrow[r, shift right] & H
\end{tikzcd}$$

Assume that the square quasi-commutes and that the right column and bottom row are exact. Then

$$i^{\vee}hg^{\vee}ff^{\vee}gh^{\vee}i = i^{\vee}hg^{\vee}gh^{\vee}i.$$

Proof. We rewrite the left-hand side of the equation successively as

$$\begin{aligned}
&i^{\vee}hh^{\vee}hg^{\vee}\,\mathrm{Ker}(E \rightrightarrows H)gh^{\vee}hh^{\vee}i \\
&\quad = i^{\vee}hg^{\vee}gh^{\vee}\,\mathrm{Ker}(G \rightrightarrows H)hg^{\vee}gh^{\vee}i \\
&\quad = i^{\vee}hg^{\vee}gh^{\vee}ii^{\vee}hg^{\vee}gh^{\vee}i \\
&\quad = i^{\vee}hg^{\vee}gh^{\vee}i
\end{aligned}$$

using the facts that the right column is exact, that the square quasi-commutes, that the bottom row is exact and that $\rho = i^{\vee}hg^{\vee}$ satisfies the Goursat condition. $\square$

We now return to Fig. 2 of Theorem 1.2 and look at the homomorphic relation

$$\varphi = kk^{\vee}ji^{\vee}hg^{\vee}fe^{\vee}e.$$

Then

$$\varphi\varphi^{\vee} = kk^{\vee}ji^{\vee}hg^{\vee}fe^{\vee}ef^{\vee}gh^{\vee}ij^{\vee}kk^{\vee}.$$

Now successively delete $e^{\vee}e$, $ff^{\vee}$, $g^{\vee}g$ and $hh^{\vee}$ by the kite Lemmas 7.1 and 7.2. Since $i^{\vee}i = 1$ and $jj^{\vee} = 1$, this leaves

(i) $\quad \varphi\varphi^{\vee} = kk^{\vee}kk^{\vee} = kk^{\vee} \le 1.$

We similarly treat

$$\varphi^{\vee}\varphi = e^{\vee}ef^{\vee}gh^{\vee}ij^{\vee}kk^{\vee}ji^{\vee}hg^{\vee}fe^{\vee}e,$$

successively deleting $kk^v, j^v j, ii^v$ and $h^v h$. Since $gg^v = 1$ and $f^v f = 1$, this leaves

(ii) $\varphi^v \varphi = e^v e e^v e = e^v e \geq 1.$

By (i) and (ii), φ is a homomorphism, hence an arrow $B \to I$ in our category. Moreover,

$$\operatorname{Ker}\varphi = \varphi^v \varphi = e^v e = \operatorname{Ker} e = \operatorname{Im}(A \rightrightarrows B),$$

$$\operatorname{Im}\varphi = \varphi\varphi^v = kk^v = \operatorname{Im} k = \operatorname{Ker}(I \rightrightarrows J),$$

and so

$$A \rightrightarrows B \to I \rightrightarrows J$$

is exact. This proves the last statement of Theorem 1.2 for Goursat$_2$ categories.

We note that the injectivity of f and the surjectivity of j are not really needed for showing that φ is a morphism. They are used in calculating the kernel and image of φ.

8. Partial equivalence relations

In this and in the next section, the word 'relation' will stand either for 'potential relation' or for 'homomorphic relation'. The results appearing in these two sections will hold in either case.

A *partial equivalence relation* or *per* on the object A of $\mathscr{C}$ is a relation $\alpha : A \nrightarrow A$ which is transitive and symmetric, but not, in general, reflexive:

$$\alpha\alpha \leq \alpha, \qquad \alpha^v \leq \alpha;$$

equivalently,

$$\alpha\alpha = \alpha, \qquad \alpha^v = \alpha.$$

We wish to view these pers as objects of a new category $\mathscr{C}^{\#}_{\mathscr{G}}$, a kind of completion of $\mathscr{C}$.

Given pers α on A and β on B, an arrow ${}_\beta\rho_\alpha : \alpha \to \beta$ will be induced by a zig-zag relation $\rho : A \nrightarrow B$ satisfying

(*) $\alpha \leq \rho^v \beta \rho, \qquad \rho\alpha\rho^v \leq \beta.$

If also ${}_\beta\sigma_\alpha : \alpha \to \beta$, we define *equality* thus:

(**) ${}_\beta\rho_\alpha = {}_\beta\sigma_\alpha \Leftrightarrow \alpha \leq \rho^v \beta\sigma \Leftrightarrow \rho\alpha\sigma^v \leq \beta.$

Note that the two inequalities on the right are equivalent, in view of (*). Indeed, assume $\alpha \leq \rho^v \beta\sigma$. Then

$$\alpha = \alpha\alpha\alpha \leq \rho^v \beta\rho\alpha\rho^v \beta\rho \leq \rho^v \beta\beta\beta\rho = \rho^v \beta\rho.$$

The converse implication is proved similarly.

Composition of arrows is defined thus:

$$({}_\gamma\sigma_\beta)({}_\beta\rho_\alpha) = {}_\gamma(\sigma\rho)_\alpha,$$

but a small argument is required to show that this is well-defined. Supposing ${}_\beta\rho_\alpha = {}_\beta\rho'_\alpha$ and ${}_\gamma\sigma_\beta = {}_\gamma\sigma'_\beta$ we must verify that ${}_\gamma(\sigma\rho)_\alpha = {}_\gamma(\sigma'\rho')_\alpha$. Indeed,

$$(\sigma\rho)\alpha(\sigma'\rho')^\upsilon = \sigma\rho\alpha\rho'^\upsilon\sigma'^\upsilon \leq \sigma\beta\sigma'^\upsilon \leq \gamma,$$

and the result follows from (**).

Finally, the *identity* arrow $1_\alpha : \alpha \to \alpha$ is defined by

$$1_\alpha = {}_\alpha(1_A)_\alpha.$$

Thus, for example,

$$({}_\beta\rho_\alpha)1_\alpha = {}_\beta(\rho 1_A)_\alpha = {}_\beta\rho_\alpha.$$

There is an obvious embedding $\mathscr{C} \to \mathscr{C}_\mathscr{G}^\#$ sending A onto 1_A and $f : A \to B$ onto ${}_{(1_B)}f_{(1_A)}$. To see that this is full and faithful, suppose the zig-zag relation $\rho : A \not\rightarrow B$ of $\mathscr{C}_\mathscr{G}$ induces an arrow $1_A \to 1_B$ in $\mathscr{C}_\mathscr{G}^\#$. Then, by (*),

$$1_A \leq \rho^\upsilon 1_B \rho, \qquad \rho 1_A \rho^\upsilon \leq 1_B.$$

But this means that ρ is a homomorphism, hence $\rho = f$ for a unique $f : A \to B$ in $\mathscr{C}$.

For what it is worth, $\mathscr{C}_\mathscr{G}^\#$ may be identified with a (non-full) subcategory of $\mathrm{Set}^\mathscr{G}$. (For the latter to be locally small, we would have to require that $\mathscr{G}$ be small.) With each per α on A we associate a functor $F_\alpha : \mathscr{G} \to$ Set defined on objects by

$$F_\alpha(G) = \mathrm{Dom}(\alpha_G)/\alpha_G.$$

Moreover, on arrows $\tau : G \to G'$, we define

$$F_\alpha(\tau)([a] \bmod \alpha_G) = [\tau(A)(g)] \bmod \alpha_{G'}$$

for any $a \in [G,A]$. Evidently, F_α is a quotient functor of a subfunctor of the representable functor $[-,A]_{\mathscr{G}}$.

Moreover, with each arrow ${}_\beta\rho_\alpha$ of $\mathscr{C}_\mathscr{G}^\#$ we associate the natural transformation $t_g : F_\alpha \to F_\beta$ defined by

$$t_\rho(G)([a] \bmod \alpha_G) = [b] \bmod \beta_G$$

whenever $b\rho_G a$. This is well-defined in view of (*). We also note that, in view of (**), ${}_\beta\rho_\alpha = {}_\beta\sigma_\alpha$ if and only if $t_\rho = t_\sigma$, since both equations translate into $\rho\alpha\rho^\upsilon \leq \beta$.

9. The two-square lemma

In a Goursat$_2$ category $\mathscr{C}_\mathscr{G}$, we have

$$\rho\rho^\upsilon\rho\rho^\upsilon = \rho\rho^\upsilon$$

for any zig-zag relation $\rho : A \not\rightarrow B$. This shows that $\alpha = \rho^{v}\rho$ and $\beta = \rho\rho^{v}$ are pers and that ${}_{\beta}\rho_{\alpha} : \alpha \to \beta$ is an isomorphism in $\mathscr{C}_{\mathscr{G}}^{\#}$, for

$$\rho\alpha\rho^{v} = \beta, \qquad \rho^{v}\beta\rho = \alpha.$$

Consider now the following diagram, in which both forks are exact:

$$\begin{tikzcd}
A \arrow[r, shift left] \arrow[r, shift right] \arrow[dr, phantom, "(1)"] & B \arrow[r, "f"] \arrow[d, "g"] \arrow[dr, phantom, "(2)"] & C \\
D \arrow[r, "h"] & E \arrow[r, shift left] \arrow[r, shift right] & F
\end{tikzcd}$$

and let $\rho_g = hh^{v}gf^{v}f : B \not\rightarrow E$. Then ρ_g induces an isomorphism $\rho_g^{v}\rho_g \to \rho_g\rho_g^{v}$ in $\mathscr{C}_{\mathscr{G}}^{\#}$. But

$$\rho_g^{v}\rho_g = f^{v}fg^{v}hh^{v}gf^{v}f = f^{v}fg^{v}\,\mathrm{Ker}(E \rightrightarrows F)gf^{v}f = \mathrm{Ker}(2),$$

say, where (2) is the open square on the right, and

$$\rho_g\rho_g^{v} = hh^{v}gf^{v}fg^{v}hh^{v} = hh^{v}g\,\mathrm{Im}(A \rightrightarrows B)g^{v}hh^{v} = \mathrm{Im}(1),$$

say, where (1) is the open square on the left.

Now close the two squares as follows:

$$\begin{tikzcd}
A \arrow[r, shift left] \arrow[r, shift right] \arrow[d, shift left] \arrow[d, shift right] \arrow[dr, phantom, "(1)"] & B \arrow[r] \arrow[d] \arrow[dr, phantom, "(2)"] & C \arrow[d, shift left] \arrow[d, shift right] \\
D \arrow[r] & E \arrow[r, shift left] \arrow[r, shift right] & F
\end{tikzcd}$$

Then (1) quasi-commutes if and only if Im(1) is symmetric about the main diagonal of the left square, and (2) quasi-commutes if and only if Ker(2) is symmetric about the main diagonal of the right square. This observation should lend support to the definition of quasi-commutativity proposed in Section 7.

We shall now take another look at Fig. 2 of Theorem 1.2. The homomorphic relations

$$\rho_g = ff^{v}gh^{v}h, \qquad \rho_h = ii^{v}hg^{v}g, \qquad \rho_i = hh^{v}ij^{v}j$$

induce the following isomorphisms, respectively:

$$\mathrm{Ker}(2) \xrightarrow{\sim} \mathrm{Im}(1), \qquad \mathrm{Ker}(2) \xrightarrow{\sim} \mathrm{Im}(3), \qquad \mathrm{Ker}(4) \xrightarrow{\sim} \mathrm{Im}(3).$$

Moreover,

$$\rho_f = gg^{v}fe^{v}e = fe^{v}e, \qquad \rho_j = kk^{v}ji^{v}i = kk^{v}j$$

induce the isomorphisms

$$e^v e \xrightarrow{\sim} \mathrm{Im}(1), \quad \mathrm{Ker}(4) \xrightarrow{\sim} kk^v.$$

Therefore,

$$\varphi^{\#} = \rho_j \rho_i^v \rho_h \rho_g^v \rho_f$$

induces the isomorphism

$$\mathrm{Im}(A \rightrightarrows B) = e^v e \xrightarrow{\sim} kk^v = \mathrm{Ker}(I \rightrightarrows J),$$

hence an arrow $1_A \to 1_B$ in $\mathscr{C}$ and therefore a morphism $A \to B$ in $\mathscr{C}$, giving an alternative proof of Theorem 1.2 for Goursat$_2$ categories.

What is the relation between $\varphi^{\#}$ and φ? In any operational category we may permute hh^v with ii^v, as we saw in Section 6. In a Maltsev$_1$ category we may also permute $h^v h$ with $g^v g$, so

$$\begin{aligned}\varphi^{\#} &= kk^v ji^v hh^v ii^v hg^v gh^v hg^v fe^v e \\ &= kk^v ji^v hg^v fe^v e = \varphi.\end{aligned}$$

It is possible that, in Goursat categories which are not Maltsev, $\varphi^{\#} \neq \varphi$ in general.

To sum up, the two-square lemma yields an alternative construction of the connecting homomorphism for Goursat categories, which coincides with our old construction in Maltsev categories. In fact, this new construction works for arbitrary operational categories: all we have to do is to redefine Im(1) and Ker(2) as the *transitive closures* of $\rho_g \rho_g^v$ and $\rho_g^v \rho_g$, respectively. This would allow us to generalize the notion of quasi-commutativity accordingly.

We note that the transitive closure of a symmetric relation is the intersection of all pers containing it. The set of relations, potential or homomorphic, between two given objects is closed under intersection.

Instead of restricting the arrows between pers to be induced by zig-zag relations, we might have admitted arbitrary homomorphic relations. This would not affect the results of Sections 8 and 9. Our choice was dictated by comparison with the established category PER of Section 13.

10. Abelian categories

What happens to our definitions and results when we restrict attention to Abelian categories? To simplify the discussion, we shall assume here that $\mathscr{C} = \mathrm{Mod}\, R$ is the category of right R-modules, R being an associative ring with unity, and $\mathscr{G} = \{G\}$, where $G = R_R$, is the ring R viewed as a right R-module. In principle, our arguments should remain valid for any Abelian category with 'enough' projectives.

To facilitate the comparison, we shall denote the usual image and kernel of $f : A \to B$ by

$$\mathrm{im}\, f = \{b \in B \mid \exists_{a \in A} b = fa\}, \qquad \ker f = \{a \in A \mid fa = 0\},$$

where $b \in B$ may stand for $b : G \to B$. It is easily seen that

$$b(\operatorname{Im} f)_G b' \Leftrightarrow b = b' \in \operatorname{im} f,$$

$$a(\operatorname{Ker} f)_G a' \Leftrightarrow a - a' \in \ker f,$$

$$b \operatorname{Im}(f,g)_G b' \Leftrightarrow b - b' \in \operatorname{im}(f - g),$$

$$a \operatorname{Ker}(f,g)_G a' \Leftrightarrow a = a' \in \ker(f - g).$$

It follows that

$$\operatorname{Im}(f,g) = \operatorname{Ker} h \Leftrightarrow \operatorname{im}(f - g) = \ker h,$$

$$\operatorname{Im} h = \operatorname{Ker}(f,g) \Leftrightarrow \operatorname{im} h = \ker(f - g).$$

Therefore,

$$A \underset{g}{\overset{f}{\rightrightarrows}} B \xrightarrow{h} C, \quad C \xrightarrow{h} B \underset{g}{\overset{f}{\rightrightarrows}} A$$

are exact if and only if

$$A \xrightarrow{f-g} B \xrightarrow{h} C, \quad C \xrightarrow{h} B \xrightarrow{f-g} A$$

are exact in the usual sense, respectively.

Proposition 10.1. *In* $\operatorname{Mod} R$, *quasi-commutativity of the squares*

$$\begin{tikzcd}
A \arrow[r, shift left, "f_1"] \arrow[r, shift right, "f_2"'] \arrow[d, shift right, "g_1"'] \arrow[d, shift left, "g_2"] & B \arrow[d, "f"] \\
C \arrow[r, "g"'] & D
\end{tikzcd}, \qquad
\begin{tikzcd}
E \arrow[r, "k"] \arrow[d, "h"'] & F \arrow[d, shift right, "k_1"'] \arrow[d, shift left, "k_2"] \\
G \arrow[r, shift left, "h_1"] \arrow[r, shift right, "h_2"'] & H
\end{tikzcd}$$

is implied by the usual commutativity:

$$f(f_1 - f_2) = g(g_1 - g_2), \quad (h_1 - h_2)h = (k_1 - k_2)k,$$

respectively.

Proof. Assume the first equation and consider the statement

(i) $d_1(f f^v g \operatorname{Im}(A \rightrightarrows C) g^v f f^v)_G d_2$

for $d_i \in D$. This asserts that there exist $b_i \in B$ and $c_i \in C$ such that

$$d_1 = fb_1 \wedge fb_1 = gc_1 \wedge c_1 - c_2 \in \operatorname{im}(g_1 - g_2) \wedge gc_2 = fb_2 \wedge fb_2 = d_2.$$

This implies that, for some $a \in A$,

$$d_1 - d_2 = fb_1 - fb_2 = gc_1 - gc_2 = g(g_1 - g_2)a = f(f_1 - f_2)a.$$

Take

$$b' = (f_1 - f_2)a - (b_1 - b_2),$$

then $fb' = 0$ and therefore

$$d_1 = gc_1 \wedge gc_1 = f(b_1 + b') \wedge (b_1 + b') - b_2 \in \mathrm{im}(f_1 - f_2) \wedge fb_2 = gc_2 \\ \wedge gc_2 = d_2.$$

Therefore,

(ii) $d_1(gg^v f \operatorname{Im}(A \rightrightarrows B) f^v gg^v)_G d_2$,

and so (i) implies (ii). The converse implication holds similarly, hence the first square quasi-commutes.

Next assume the second equation and consider the statement

(iii) $e_1(h^v hk^v \operatorname{Ker}(F \rightrightarrows H) kh^v h)_G e_2$

for $e_i \in E$. This asserts that there exist $e_i' \in E$ such that

$$he_1 = he_1' \wedge ke_1' = ke_2' \in \ker(k_1 - k_2) \wedge he_2' = he_2.$$

This implies that

$$e_1', e_2' \in \ker((k_1 - k)k) = \ker((h_1 - h_2)h),$$

hence that

$$he = he_1' \wedge he_2 = he_2' \in \ker(h_1 - h_2).$$

Take $e = e_1 + e_2' - e_1'$ and $e' = e_2$, then

$$ke_1 = ke \wedge he = he' \in \ker(h_1 - h_2) \wedge ke' = ke_2,$$

so that

(iv) $e_1(k^v kh^v \operatorname{Ker}(G \rightrightarrows H) hk^v k)_G e_2$.

Therefore (iii) implies (iv). The converse implication holds similarly, hence the second square quasi-commutes.

It follows that our construction of the connecting homomorphism implies the usual one in a module category. $\square$

11. Algebraic categories

A single-sorted algebraic category $\mathscr{C}$ such as $\operatorname{Mod} R$ may be described classically as having as objects sets with finitary operations satisfying certain equations and as arrows mappings which preserve these operations. (In case of Mod R, the elements of R are to be counted among the operations.)

If we denote by G the forgetful functor from $\mathscr{C}$ to Set, n-ary operations may be viewed as natural transformations $G^n \to G$. Now G is represented by the free algebra in one generator, according to our convention also denoted by G, and the n-ary operation then becomes $G \to nG$ in $\mathscr{C}$.

A single-sorted algebraic category is operational in our sense, with $\mathscr{G} = \{G\}$, where G is projective, hence regular epis are the same as surjections. In such a category, there is a one-to-one correspondence between homomorphic relations $\rho : A \not\rightarrow B$ and subalgebras $R \subseteq B \times A$. It follows that every homomorphic relation is representable: $\rho = gf^v$, where $f = pm$ and $g = qm$, $m : R \to B \times A$ being the inclusion and $p : B \times A \to A$ and $q : B \times A \to B$ the canonical projections.

These single-sorted finitary algebraic categories may be generalized in two directions. By removing the restriction that all operations be finitary, we get Linton's equational and varietal categories, e.g. the category of compact Hausdorff spaces. (In varietal categories G is representable, hence may be taken in $\mathscr{C}^{\mathrm{op}}$, not so in equational categories.) By removing the restriction that there is only a single sort, we obtain the multi-sorted algebraic categories of Higgins, e.g. the category Mod with two sorts: one for rings and one for Abelian groups.

All these generalized algebraic categories are operational and so are full subcategories of such; to mention only one example: the category of normed vector spaces with norm decreasing linear mappings as morphisms.

12. Operational categories simplified

Let $\mathscr{G}$ be a full subcategory of $\mathrm{Set}^{\mathscr{C}}$ and assume that $\mathscr{G}$ is closed under products. Then an operation $\omega : P \to G$, with $P = \prod_{i \in I} G_i$, is simply an arrow in $\mathscr{G}$. What does it mean for a potential morphism $\varphi : A \to B$ to preserve the operation ω? In Section 2 we expressed this by the commutativity of a square, to wit, by the equation

$$\varphi_G \omega(A) = \omega(B) \varphi_P,$$

where however φ_P was defined by the universal property of the product P. Now, as P is in $\mathscr{G}$, φ_P may be identified with the value of the natural transformation φ at P, as anticipated by the notation. But then the square commutes merely by virtue of φ being a natural transformation. We thus have:

Proposition 12.1. *If $\mathscr{G}$ is a full subcategory of* Set$^{\mathscr{C}}$ *closed under products, then any potential morphism $\varphi : A \to B$ is a homomorphism.*

Postulates I and II now assert that every potential morphism is a morphism, uniquely represented by an arrow $f : A \to B$ of $\mathscr{C}$. In case $\mathscr{G} \subseteq \mathscr{C}^{\mathrm{op}}$, $\mathscr{G}$ is then what Isbell calls *adequate* and $\mathscr{C}$ may be viewed as a full subcategory of $\mathrm{Set}^{\mathscr{G}}$. Moreover, every object

C of $\mathscr{C}$ may be viewed as a product preserving functor $\mathscr{G} \to \text{Set}$, as in an algebraic category à la Lawvere.

Let us now return to the situation of Section 2, where $\mathscr{G}$ is no longer assumed to be closed under products, and let $\mathscr{F}$ be its product closure in $\text{Set}^{\mathscr{C}}$.

Proposition 12.2. *Let $\mathscr{G}$ be a subcategory of $\text{Set}^{\mathscr{C}}$, $\mathscr{F}$ its full product closure. Then every homomorphism $\varphi : A \to B$ in $\mathscr{C}_{\mathscr{G}}$ extends to a unique homomorphism $\psi : A \to B$ in $\mathscr{C}_{\mathscr{F}}$ and, conversely, every such ψ restricts to a unique φ.*

Proof. Let $F = \prod_{i \in I} G_i$ with canonical projections $\pi_i : F \to G_i$. Define the natural transformation

$$\psi : [_{\mathscr{F}}, A] \to [_{\mathscr{F}}, B];$$

thus

$$\psi_F(a) = (b) \Leftrightarrow \forall_{i \in I} \varphi_{G_i}(a_i) = b_i,$$

where $a \in F(A)$, $b \in F(B)$, $a_i = \pi_i(A)(a)$ and $b_i = \pi_i(B)(b)$. It is easily verified that ψ is indeed a natural transformation and we claim that it extends φ.

For suppose F is already in $\mathscr{G}$, then the natural transformation

$$1_F : \prod_{i \in I} G_i \to F$$

is an operation, hence

$$\forall_{i \in I} \varphi_{G_i}(a_i) = b_i \Rightarrow \varphi_F(a) = b,$$

by the definition of homomorphism in Section 2. The converse implication holds because φ is natural. Therefore $\varphi_F = \psi_F$.

On the other hand, any homomorphism of $\mathscr{C}_{\mathscr{F}}$ clearly restricts to a homomorphism of $\mathscr{C}_{\mathscr{G}}$. $\square$

A result analogous to Proposition 12.2 holds for homomorphic relations, but we will not spell it out.

13. Recursively enumerable relations

Let us review some facts about ordinary binary relations $\rho : \mathbb{N} \not\rightarrow \mathbb{N}$ whose graphs are recursively enumerable, call them *recursively enumerable relations* for short. Clearly, every such relation has the form $\rho = fg^{\cup}$, that is to say,

$$\forall_{m,n \in \mathbb{N}} (n\rho m \Leftrightarrow \exists_{k \in \mathbb{N}} (n = f(k) \wedge m = g(k)))$$

where f and g are (general) recursive functions in one variable. (In fact, by a result of Rosser's they may be taken to be primitive recursive.)

The set of recursively enumerable relations $\mathbb{N} \not\rightarrow \mathbb{N}$ is closed under composition of relations. For, if the graph of $g^v h$ is recursive, it is recursively enumerable, hence $g^v h = pq^v$, for some (general or primitive) recursive functions $p, q : \mathbb{N} \to \mathbb{N}$.
Therefore,

$$(fg^v)(hk^v) = fpq^v k^v = (fp)(kq)^v$$

is again recursively enumerable.

Note that fg^v is a *partial recursive function* if it is single-valued, i.e.,

$$fg^v g f^v = (fg^v)(fg^v)^v \leq 1_{\mathbb{N}},$$

which is equivalent to $g^v g \leq f^v f$. Note also that the partial recursive function fg^v is a general recursive function if and only if it is universally defined, which is easily seen to be the case if and only if g is surjective in the usual sense.

The category PER plays a role in theoretical computer science as a model of polymorphic lambda-calculus. We shall now present a new construction of PER. Its objects are partial equivalence relations on $\mathbb{N}$ and its morphisms ${}_\beta\rho_\alpha : \alpha \to \beta$ are (equivalence classes of) recursively enumerable relations $\rho : \mathbb{N} \not\rightarrow \mathbb{N}$ such that $\alpha \leq \rho^v \beta \rho$ and $\rho\alpha\rho^v \leq \beta$. Equality between morphisms $\alpha \to \beta$ is defined thus: ${}_\beta\rho_\alpha = {}_\beta\sigma_\alpha$ provided $\alpha \leq \rho^v \beta \sigma$ or, equivalently, $\rho\alpha\sigma^v \leq \beta$.

This is not the usual construction, which requires that ρ be a partial recursive function, in which case the condition $\rho\alpha\rho^v \leq \beta$ is deducible from $\alpha \leq \rho^v\beta\rho$. However, it is equivalent to the usual construction, since ${}_\beta\rho_\alpha = {}_\beta\rho^*_\alpha$, where ρ^* is the partial recursive function defined as follows:

$$\rho^*(m) = \text{smallest } n \text{ such that } n\rho m.$$

Proposition 13.1. *If ${}_\beta\rho_\alpha$ is a morphism in PER, as defined above, then so is ${}_\beta\rho^*_\alpha$ and ${}_\beta\rho^*_\alpha = {}_\beta\rho_\alpha$.*

Proof. We are given that $\rho\alpha\rho^v \leq \beta$ and $\alpha \leq \rho^v\beta\rho$ and wish to show the same with ρ replaced by ρ^*. Since $\rho^* \leq \rho$, we immediately obtain $\rho^*\alpha\rho^{*v} \leq \rho\alpha\rho^v \leq \beta$. But why is $\alpha \leq \rho^{*v}\alpha\rho^*$?

Suppose $n\alpha m$, hence $n(\rho^v\beta\rho)m$, i.e.,

$$\exists_{k,\ell\in\mathbb{N}}(k\rho n \wedge k\beta\ell \wedge \ell\rho m).$$

Let $k' = \rho^*(n)$ and $\ell' = \rho^*(m)$. Then

$$k'\rho n \wedge n\alpha m \wedge \ell'\rho m,$$

hence $k'(\rho\alpha\rho^v)\ell'$ and so $k'\beta\ell'$. Thus,

$$k'\rho^* n \wedge k'\beta\ell' \wedge \ell'\rho^* m,$$

hence $n(\rho^{*v}\beta\rho^*)m$. This shows that $\alpha \leq \rho^{*v}\beta\rho^*$, as required.

Finally, we have

$$\rho^* \alpha \rho^v \le \rho \alpha \rho^v \le \beta.$$

hence ${}_\beta\rho^*_\alpha = {}_\beta\rho_\alpha$. $\square$

There is a resemblance between our new construction of PER and the construction of $\mathscr{C}^\#_\mathscr{G}$ in Section 8. Can the relation between the two constructions be made precise?

Let $\mathscr{C}$ be the monoid of general recursive functions $\mathbb{N} \to \mathbb{N}$ and take $\mathscr{G} = \mathscr{C}^{\mathrm{op}}$. Then $\mathscr{G}$ and $\mathscr{C}$ have only one object, namely $\mathbb{N}$. In particular, $\mathscr{C} = \mathrm{Hom}(\mathbb{N}, \mathbb{N})$. With any relation ρ on $\mathbb{N}$ we associate the relation $\rho^\dagger_\mathbb{N}$ in $\mathscr{C}_\mathscr{G}$ as follows:

$$b\rho^\dagger_\mathbb{N} a \Leftrightarrow \forall_{n\in\mathbb{N}} b(n)\rho a(n).$$

Now the expression on the right implies

$$\forall_{c\,:\,\mathbb{N}\to\mathbb{N}} \forall_{k\in\mathbb{N}} b(c(k))\rho a(c(k)),$$

i.e.,

$$\forall_{c\,:\,\mathbb{N}\to\mathbb{N}} (bc)\rho^\dagger_\mathbb{N}(ac).$$

Thus, $\{\rho^\dagger_\mathbb{N}\}$ is what we have called a natural family of relations (with one member), making $\rho^\dagger$ a potential relation in $\mathscr{C}_\mathscr{G}$.

We note that

$$c(\sigma\rho)^\dagger_\mathbb{N} a \Leftrightarrow \forall_{n\in\mathbb{N}} c(n)(\sigma\rho)a(n)$$

$$\Leftrightarrow \forall_{n\in\mathbb{N}} \exists_{k\in\mathbb{N}} (c(n)\sigma k \wedge k\rho a(n)).$$

Taking

$$b(n) = \text{ smallest } k \text{ such that } (c(n)\sigma k \wedge k\rho a(n)),$$

we see that the above holds

$$\Leftrightarrow \exists_{b\,:\,\mathbb{N}\to\mathbb{N}} \forall_{n\in\mathbb{N}} (c(n)\sigma b(n) \wedge b(n)\rho a(n))$$

$$\Leftrightarrow \exists_{b\,:\,\mathbb{N}\to\mathbb{N}} (c\sigma^\dagger_\mathbb{N} b \wedge b\rho^\dagger_\mathbb{N} a)$$

$$\Leftrightarrow c\sigma^\dagger_\mathbb{N}\rho^\dagger_\mathbb{N} a$$

$$\Leftrightarrow c(\sigma^\dagger\rho^\dagger)_\mathbb{N} a,$$

hence

$$(\sigma\rho)^\dagger = \sigma^\dagger\rho^\dagger.$$

Note that this argument makes use of the *minimization scheme*, which would not have worked if we had taken $\mathscr{C}$ to be the monoid of primitive recursive functions.

Thus, $(-)^\dagger$ is a homomorphism of the monoid of binary relations on $\mathbb{N}$ to the monoid of potential relations in $\mathscr{C}_\mathscr{G}$. Moreover, it is easily seen to preserve the converse operation and the partial order of relations:

$$\rho^{\dagger v} = \rho^{v\dagger}, \quad \rho \le \sigma \Rightarrow \rho^\dagger \le \sigma^\dagger.$$

Let us investigate what it does to recursively enumerable relations.

We observe that

$$(fg^v)^\dagger = f^\dagger g^{\dagger v}.$$

Now

$$\begin{aligned} bf^\dagger_\mathbb{N} a &\Leftrightarrow \forall_{n\in\mathbb{N}} b(n) = f(a(n)) \\ &\Leftrightarrow b = fa \\ &\Leftrightarrow bf_\mathbb{N} a, \end{aligned}$$

according to our convention of writing f for $[_{\overline{\mathscr{G}}}, f]$. Therefore $f^\dagger = f$, and so

$$(fg^v)^\dagger = fg^v;$$

in other words, if ρ is any recursively enumerable relation fg^v, $\rho^\dagger$ is the representable homomorphic relation also denoted by fg^v.

Conversely, let there be given a potential relation π in $\mathscr{C}_\mathscr{G}$, that is, a natural family $\{\pi_\mathbb{N}\}$ of one member. We define the relation $\pi^\S$ on $\mathbb{N}$ thus:

$$n\pi^\S m \Leftrightarrow c_n \pi_\mathbb{N} c_m,$$

where c_n is the function $\mathbb{N} \to \mathbb{N}$ with constant value $n \in \mathbb{N}$.

We claim that

$$\rho^{\dagger\S} = \rho, \qquad \pi^{\S\dagger} = \pi.$$

Indeed,

$$\begin{aligned} n\rho^{\dagger\S} m &\Leftrightarrow c_n \rho^\dagger_\mathbb{N} c_m \\ &\Leftrightarrow \forall_{k\in\mathbb{N}} c_n(k) \rho c_m(k) \\ &\Leftrightarrow n\rho m \end{aligned}$$

and

$$\begin{aligned} b\pi^{\S\dagger}_\mathbb{N} a &\Leftrightarrow \forall_{k\in\mathbb{N}} b(k) \pi^\S a(k) \\ &\Leftrightarrow \forall_{k\in\mathbb{N}} c_{b(k)} \pi_\mathbb{N} c_{a(k)} \\ &\Leftrightarrow \forall_{k\in\mathbb{N}} (bc_k) \pi_\mathbb{N} (ac_k) \\ &\Leftrightarrow b\pi_\mathbb{N} a, \end{aligned}$$

since

$$c_{b(k)}(n) = b(k) = b(c_k(n)) = (bc_k)(n),$$

and by naturality of $\{\pi_{\mathbb{N}}\}$.

We have thus proved:

Proposition 13.2. *There is an isomorphism between the monoid of binary relations on* $\mathbb{N}$ *and the monoid of potential relations on* $\mathscr{C}_{\mathscr{G}}$, *where* $\mathscr{C}$ *is the monoid of recursive functions* $\mathbb{N} \to \mathbb{N}$ *and* $\mathscr{G} = \mathscr{C}^{\mathrm{op}}$, *which preserves the converse operation and the partial order. Moreover, it induces an isomorphism between the monoid of recursively enumerable relations on* $\mathbb{N}$ *and the monoid of zig-zag homomorphic relations in* $\mathscr{C}_{\mathscr{G}}$.

We are now in a position to compare the categories PER and $\mathscr{C}^{\#}_{\mathscr{G}}$, provided the 'relations' of Section 8 are taken to be 'potential relations'.

Corollary 13.3. *If* $\mathscr{C}$ *is the monoid of recursive functions* $\mathbb{N} \to \mathbb{N}$ *and* $\mathscr{G} = \mathscr{C}^{\mathrm{op}}$, $\mathscr{C}^{\#}_{\mathscr{G}}$ *may be identified with* PER.

This observation is not really surprising, as it influenced our choice of the objects and arrows for $\mathscr{C}^{\#}_{\mathscr{G}}$. Otherwise we might have allowed its arrows to be induced by all homomorphic relations, not just zig-zag ones. We might have admitted as objects only homomorphic pers, but then we would not have been sure that the constructions of PER in computer science is a special case of that of $\mathscr{C}^{\#}_{\mathscr{G}}$ in algebra.

14. Commentary

This article might have been called 'two constructions in search of suitable definitions'. It had been clear to me for some time that the usual constructions of the so-called connecting homomorphism in homological algebra should work for arbitrary Maltsev and Goursat categories. The problem was to find the right definitions for 'exactness' '(quasi-) commutativity', etc. Having produced too stringent definitions in earlier attempts, I finally straightened things out for algebraic categories in my contribution to the Magari conference of 1994 and am here considering more general operational categories, which include ordinary locally small categories as a special case.

It turns out that there are, in fact, two constructions of the connecting homomorphism, one as a straightforward zig-zag relation and the other as an application of the 'two-square lemma'. These two constructions agree in Maltsev categories, but seem to be different in Goursat categories. Moreover, the second construction generalizes to arbitrary operational categories, provided one is willing to make some concessions.

Here follow some comments on the separate sections of this article.

1. It has been known for a long time that the connecting homomorphism, as in Theorem 1.1, can be constructed with the help of binary relations; see [28, 29, Ch. II, 6] for the early history.

A generalization as in Theorem 1.2 had been obtained for Maltsev varieties in [19] and for Maltsev categories in [6]. Unfortunately, both these versions suffered from the choice of bad definitions for exactness and commutativity, as pointed out in [5]. For Goursat varieties, this was finally rectified in [22].

2. The 'operational categories' discussed here are more general than those of Wyler [35] and those of Lambek and Rattray [23], but less general than those of Jay [13] which I had proposed at the Midwest Category Seminar at Waterloo in 1968. I had a hard time deciding on the right definition in the present context. For most purposes it would have been sufficient to take $\mathscr{G} \subseteq \mathscr{C}^{\mathrm{op}}$, all sorts being representable, but this would have eliminated Linton's [26] equational categories and the multi-sorted algebras of Higgins [10]. Postulates I and II for operational categories are introduced for convenience only, to facilitate the statement of results. We could have avoided these postulates, provided we had employed a somewhat more cumbersome language, distinguishing between homomorphisms, morphisms and arrows in $\mathscr{C}$.

3. Homomorphic relations in algebraic categories had been studied in [16] and relations in Abelian categories in [11]. Relations in regular categories were investigated e.g. in [1, 3, 8, 30, 31, 34] and, in connection with Maltsev and Goursat conditions, in [6]. I am distinguishing between *potential* and *homomorphic* relations, not having been able to decide which of the two should just be called *relations*.

4. Injections and surjections in operational categories resemble those in algebraic varieties and, in general, must be distinguished from monomorphism and epimorphisms respectively.

5. The notion of exactness discussed here depends on the correct definition of $\mathrm{Im}(f, g)$ as a congruence relation and not as $fg^{\vee}$, as had been the case in the earlier publications mentioned in the comments on Section 1.

6. I had noticed in [16] that certain key results of group theory were really valid in Maltsev varieties. It has now become clear, as was pointed out to me by Carboni and Pedicchio, that they were valid in Goursat varieties. I had already learned of a crucial result by Goursat [9] from H.S.M. Coxeter and put Goursat's name in the titles of two of my early papers.

Natural examples of Goursat varieties that are not Maltsev are hard to come by; the first such example was pointed out in [33], as I learned from K. Denecke. Both Maltsev and Goursat varieties can be characterized in a number of equivalent ways. In generalizing these notions to arbitrary operational categories, I could not decide on the best definition, hence the subscript 1, 2 or 3 on 'Maltsev' and 'Goursat'. Of course, in categories in which all homomorphic relations are representable, such as the single-sorted algebraic categories of Section 11, the subscripts are not necessary.

7. There was a problem to find the appropriate notion of commutativity for squares in which some arrows are doubled. As mentioned in the comments to Section 1, the

definition picked earlier was too restrictive. The notion of quasi-commutativity proposed here specializes to that in my Magari paper; it works in the present context, but may not be the final one. The kite lemmas were introduced in answer to a suggestion by Robert Seely.

8. The category $\mathscr{C}^{\#}_{\mathscr{G}}$ discussed here is only one of a number of similar categories. In [20, 21] two such categories were considered, both with $\mathscr{G} = \mathscr{C}^{\mathrm{op}}$. In both the objects were not homomorphic pers, but potential pers. In $\mathscr{C}^{R}$ the arrows were induced by arrows of $\mathscr{C}$ and in $\mathscr{C}^{(R)}$ the arrows were induced by potential relations. $\mathscr{C}^{(R)}$ is a full subcategory of the category of all functors $\mathscr{C}^{\mathrm{op}} \to \mathrm{Set}$.

9. The two-square lemma plays a crucial role for modules in my book of 1966. I had generalized it to groups in [17], probably with too stringent a notion of exactness. It was generalized to Maltsev varieties in [19] and to Maltsev categories in [6], subject to the defects already mentioned in Section 1, and to Goursat varieties in [22]. In these categories it was assumed that $\mathscr{G} = \mathscr{C}^{\mathrm{op}}$.

10. At least for module categories, the present constructions reduce to the usual ones. To extend them to arbitrary Abelian (and even more general) categories, some more work has to be done, see e.g. [25].

11. The homomorphic relations studied here agree with those of Lambek [16] in algebraic varieties, the forgetful functor to sets being representable. It is not representable in Linton's [26] varietal categories, nor presumably in the many-sorted algebraic varieties of Higgins [10]. These were taken up again by Birkhoff and Lipson [2] and probably by any number of categorists following Lawvere [24]. Full subcategories of varietal categories were studied in [23] under the name of 'operational categories'.

12. The definitions of operational categories and homomorphic relations may be much simplified if it is assumed that the class of sorts is closed under arbitrary products. This definition was not adopted at the beginning here, to mention only one reason, because even in algebraic categories there is usually given only a set of sorts closed under finite products.

13. Evidently, the category PER of theoretical computer science [27] should be a special case of $\mathscr{C}^{\#}_{\mathscr{G}}$. In this section, we achieve this by adjusting the definition of the latter category, over which we have control, and by showing that the usual definition of PER can be recast to suit our present purpose, following a suggestion by Michael Makkai. I have chosen 'relation' to mean 'potential relation', but it is conceivable that, in the operational category $\mathscr{C}_{\mathscr{G}}$ considered here, all potential relations are homomorphic, a possibility I have not investigated.

Postscript

If one wishes to answer all the questions that arise from a paper, it will never get written. In particular, I defer the following:

(i) to compare the present definition of homomorphic relations with the usual definition of relations in regular or exact categories;

(ii) to consider Abelian categories other than Mod R;

(iii) to investigate the universal property of the embedding $\mathscr{C}_{\mathscr{G}} \to \mathscr{C}^{\#}_{\mathscr{G}}$ and to determine when it is an equivalence;

(iv) to see whether Hyland's realizability topos can be obtained by an analogous construction (see some suggestive similarities in [4]).

Finally, let me confess that I still have some misgivings about the scope of the present article. I waver between thinking it is too general and not general enough.

Acknowledgements

Support from the Natural Sciences and Engineering Research Council is herewith acknowledged. The author is grateful to Michael Makkai and Robert Seely for asking the right questions.

References

[1] M. Barr, P.A. Grillet and D.H. van Osdal, Exact categories and categories of sheaves, Lecture Notes in Math., Vol. 236 (Springer, Berlin, 1971).
[2] G. Birkhoff and J.D. Lipson, Heterogeneous algebras, J. Combin. Theory 8 (1970) 115–133.
[3] M.S. Calenko, V.B. Gisin and D.A. Raikov, Ordered categories with involution, Dissertationes Matematicae (=Rozprawy Matematyczne) 227 (1984) 1–111.
[4] A. Carboni, P.J. Freyd and A. Scedrov, A categorical approach to realizability and polymorphic types, Lecture Notes in Computer Science, Vol. 298 (Springer, Berlin, 1988) 23–42.
[5] A. Carboni, G.M. Kelly and M.C. Pedicchio, Some remarks on Maltsev and Goursat categories, Appl. Categorical Struct. 1 (1993) 385–421.
[6] A. Carboni, J. Lambek and M.C. Pedicchio, Diagram chasing in Mal'cev categories, J. Pure Appl. Algebra 69 (1990) 271–284.
[7] P.J. Freyd, Abelian Categories (Harper and Row, New York, 1964).
[8] P.J. Freyd and A. Scedrov, Categories, Allegories (North-Holland, Amsterdam, 1990).
[9] É. Goursat, Sur les substitutions orthogonales, Ann. Sci. École Normale Supérieure (3) 6 (1889) 9–102.
[10] P.J. Higgins, Algebras with a scheme of operators, Math. Nachrichten 27 (1963) 115–132.
[11] P. Hilton, Correspondences and exact sequences, in: S. Eilenberg et al., eds., Proc. Conf. on Categorical Algebra, La Jolla, 1965 (Springer, New York, 1966).
[12] J.R. Isbell, Subobjects, adequacy, completeness and categories of algebras, Rozprawy Matematyczne 36 (1964) 1–32.
[13] C.B. Jay, Generalizing the structure-semantics adjunction: operational categories, Thesis, McGill University, 1984.
[14] C.B. Jay, Lambek's operational categories, Bull. Australian Math. Soc. 33 (1986) 161–176.
[15] S.C. Kleene, Introduction to Metamathematics (Van Nostrand, New York, 1952).
[16] J. Lambek, Goursat's theorem and the Zassenhaus lemma, Can. J. Math. 10 (1957) 45–56.
[17] J. Lambek, Goursat's theorem and homological algebra, Can. Math. Bull. 7 (1964) 597–608.
[18] J. Lambek, Lectures on Rings and Modules (Blaisdell, Waltham, 1966; Chelsea, New York, 1986).
[19] J. Lambek, On the ubiquity of Mal'cev operations, Contemp. Math. 131 (1992) 135–146.
[20] J. Lambek, Least fixpoints of endofuctors of cartesian closed categories, Math. Struct. Comput. Sci. 3 (1993) 229–257.
[21] J. Lambek, Some aspects of categorical logic, in: D. Prawitz et al., eds., Logic, Methodology and Philosophy of Science IX (Elsevier, Amsterdam, 1994).
[22] J. Lambek, The butterfly and the serpent, in: P. Agliano and A. Ursini, eds., Logic and Algebra, Proc. 1994 Magiari Conf., 1995.
[23] J. Lambek and B.A. Rattray, A general Stone–Gelfand duality, Trans. Amer. Math. Soc. 248 (1979) 1–35.

[24] F.W. Lawvere, Functorial semantics of algebraic theories, Thesis, Columbia University, 1963.
[25] J.B. Leicht, Axiomatic proof of J. Lambek's homological theorem, Can. Math. Bull. 7 (1964) 609–613.
[26] F.E.J. Linton, Some aspects of equational systems, in: S. Eilenberg et al., eds., Proc. Conf. on Categorical Algebra, La Jolla, 1965 (Springer, New York, 1966).
[27] G. Longo and E. Moggi, Constructive natural deduction and its "ω-set" interpretation, Math. Struct. Comput. Sci. 1 (1991) 215–254.
[28] P. Lorenzen, Über die Korrespondenzen einer Struktur, Math. Z. 60 (1954).
[29] S. MacLane, Homology (Springer, New York, 1967).
[30] J. Meisen, Relations in categories, Thesis, McGill University, 1972.
[31] J. Meisen, On bicategories of relations and pullback spans, Comm. Algebra 1 (1974) 377–401.
[32] J. Meisen, Relations in regular categories, Lecture Notes in Math., Vol. 418 (Springer, Berlin, 1974) 90–92.
[33] A. Mitschke, Implication algebras are 3-permutable and 3-distributive, Algebra Universalis 1 (1971) 182–186.
[34] A. Möbus, Relational-Algebren, Inaugural Dissertion, Universität Düsseldorf, 1981.
[35] O. Wyler, Operational categories, in: S. Eilenberg et al., eds., Proc. Conf. on Categorical Algebra, La Jolla, 1965 (Springer, New York, 1966).